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# STRESSES IN A BIELASTIC LAYER WITH A SHEAR EDGE CRACK*

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The distribution of the displacements and stresses in a bielastic layer of finite thickness is studied when one monolayer is torn completely by a longitudinal shear edge crack perpendicular to the layer boundary, and with its tip at the interface of the two media. Arbitrary symmetrically distributed stresses are given on the crack edges, the bielastic layer surface is free of external loads. There are no displacements and stresses within the layer at infinity. The whole structure is symmetrical about the plane of the crack and the displacement outside the crack in this plane is assumed to be zero. An asymptotic form is obtained for the stress field near the crack apex and an expression is found for the stress intensity factor $k_{III}$ in terms of an auxiliary unknown function for which the nature of the singularity is established and an integral equation with extracted singular part is obtained. The Keldysh-Sedov method is used to solve the auxiliary boundary value problems.

1. *Formulation of the problem and method of solution.* Let two elastic homogeneous isotropic strips $\Pi_1$: $0 < x < h_1$, $|y| < \infty$ and $\Pi_2$: $h_1 < x < h_1 + h_2 = H$, $|y| < \infty$ consist of two, generally different, materials with shear moduli $\mu_1$ and $\mu_2$, respectively, where the strip $\Pi_1$ is torn completely by a longitudinal shear edge crack $\gamma$: $y = 0$, $0 \leqslant x \leqslant h_1$. The displacement vector components in the strips $\Pi_j$ in the antiplane strain case under consideration have the form

$$W_x^{(j)} = W_y^{(j)} = 0,\ W_z^{(j)} = W(x,\ y)\ \ (j = 1,\ 2)$$

The stress tensor components are given by the formula

$$\sigma_x^{(j)} = \sigma_y^{(j)} = \sigma_z^{(j)} = \tau_{xy}^{(j)} = 0,\quad \tau_{xz}^{(j)} = \mu_j \frac{\partial W^{(j)}}{\partial x},\quad \tau_{yz}^{(j)} = \mu_j \frac{\partial W^{(j)}}{\partial y}$$

We shall consider symmetrically distributed stresses

$$\mu_1 \frac{\partial W^{(1)}}{\partial y}(x, \pm 0) = -\sigma_1(x)\quad (0 \leqslant x \leqslant h_1)$$

applied to the slit edges $\gamma$, where the given function $\sigma_1(x)$ is continuous in the segment $0 \leqslant x \leqslant h_1$ and satisfies the Lipschitz condition of arbitrarily small positive order at the point $x = h_1 - 0$.

Along the common boundary $x = h_1$, $|y| < \infty$, the strips $\Pi_1, \Pi_2$ adhere rigidly

$$W^{(1)}(h_1 - 0,\ y) = W^{(2)}(h_1 + 0,\ y)\ \ (|y| < \infty) \tag{1.1}$$

$$\mu_1 \frac{\partial W^{(1)}}{\partial x}(h_1 - 0, y) = \mu_2 \frac{\partial W^{(2)}}{\partial x}(h_1 = 0, y)\quad (|y| < \infty) \tag{1.2}$$

and the boundary $\partial\Pi$ of the bielastic strip $\Pi = \Pi_1 \cup \Pi_2$ is free from external loads, i.e.,

$$\frac{\partial W^{(1)}}{\partial x}(+0, y) = \frac{\partial W^{(2)}}{\partial x}(H - 0, y) = 0\quad (|y| < \infty)$$

By virtue of the existing symmetry about the $y = 0$ axis, the corresponding half-strips

$$\Pi_1^+ : 0 < x < h_1,\ y > 0;\ \Pi_2^+ : h_1 < x < h_1 + h_2 = H,\ y > 0$$

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can be considered in place of the strips $\Pi_j$ and we can put

$$W^{(2)}(x, +0) = 0 \quad (h_1 \leqslant x \leqslant H)$$

We also assume that the function $W^{(j)}(x, y)$ are continuous together with the first partial derivatives up to the boundaries $\partial\Pi_j^+$ of the appropriate half-strip $\Pi_j^+$ with the exception of the tip $(h_1, 0)$ of the crack $\gamma$, where these derivatives can have an integrable singularity. The functions $W^{(j)}(x, y)$ together with their first partial derivatives vanish (Fig.1) at infinity in the corresponding half-strips.

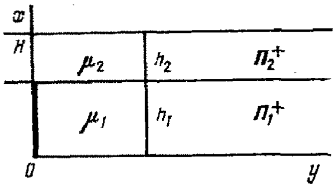


Fig.1

To will seek the objective function

$$\sigma(x) = \tau_{yz}^{(2)}(x, +0) = \mu_2 \frac{\partial W^{(2)}}{\partial y}(x, +0) \quad (h_1 < x \leqslant H) \tag{1.3}$$

we take account of (1.2) and introduce the auxiliary unknown function

$$\varphi(y) = \frac{\partial W^{(1)}}{\partial x}(h_1 - 0, y) = \frac{1}{\omega} \frac{\partial W^{(2)}}{\partial x}(h_1 + 0, y) \quad (y > 0), \ \omega = \mu_1/\mu_2 \tag{1.4}$$

which is continuous for $0 < y < \infty$ by assumption. In addition we assume that

$$\varphi(y) = O(y^{-2-\varepsilon}) \ (y \to +\infty), \ \varepsilon > 0 \tag{1.5}$$

$$\exists\alpha \in ]0, 1[, \quad \exists y_0 > 0 : \varphi(y) = y^{\alpha-1}(a_0 + a_1 y + \ldots) + (b_0 + b_1 y + \ldots)$$
$$(0 < y \leqslant y_0)$$

In particular, thereby

$$\varphi(y) = O(1) \quad (y \geqslant y_0), \ \int_0^\infty |\varphi(t)|\, dt < \infty, \quad \int_0^\infty t\,|\varphi(t)|\, dt < \infty \tag{1.6}$$

Now we have the following boundary value problems for the functions $W^{(j)}(x, y)$

$$\nabla_{xy}^2 W^{(1)} = 0, \ ((x, y) \in \Pi_1^+)$$
$$\frac{\partial W^{(1)}}{\partial x}(+0, y) = 0, \ \frac{\partial W^{(1)}}{\partial x}(h_1 - 0, y) = \varphi(y) \quad (y > 0) \tag{1.7}$$
$$\frac{\partial W^{(1)}}{\partial y}(x, +0) = -\left(\frac{1}{\mu_1}\right)\sigma_1(x) \quad (0 \leqslant x \leqslant h_1)$$

$$|W^{(1)}| + |\operatorname{grad} W^{(1)}| = o(1) \quad ((x, y) \to \infty, \ (x, y) \in \Pi_1^+) \tag{1.8}$$

$$\nabla_{xy}^2 W^{(2)} = 0, \quad ((x, y) \in \Pi_2^+)$$
$$\frac{1}{\omega} \frac{\partial W^{(2)}}{\partial x}(h_1 + 0, y) = \varphi(y), \quad \frac{\partial W^{(2)}}{\partial x}(H - 0, y) = 0 \quad (y > 0), \ \omega = \mu_1/\mu_2 \tag{1.9}$$
$$W^{(2)}(x, +0) = 0 \quad (h_1 \leqslant x \leqslant H)$$

$$|W^{(2)}| + |\operatorname{grad} W^{(2)}| = o(1) \ ((x, y) \to \infty, \ (x, y) \in \Pi_2^+) \tag{1.10}$$

Solving these problems we find the objective function $\sigma(x)$ by means of (1.3) and by using the identity (1.1) we obtain an integral equation for the auxiliary unknown function $\varphi(y)$.

**2.** ***Solution of the boundary value problem (1.7) and (1.8).*** Let $S(x, y)$ be some solution of Problem (1.7) and (1.8). Then a function $T(x, y$ can be found according to (1.7) such that

$$f(z) = S(x, y) + iT(x, y) \in A(\Pi_1^+) \ (z = x + iy)$$

i.e., the function $f(z)$ is regular in the domain $\Pi_1^+$ of the complex plane $C_z$. We will consider the function $w(z) = -\cos(\pi z/h_1)$. The domain $\Pi_1^+$ of this function is univalent and maps conformally into the half-plane $\operatorname{Im} w > 0$ of the complex plane $C_w$. Using the notation $F(w) = f(z(w))$ $(\operatorname{Im} w > 0)$, we find by virtue of Condition (1.8) and the first of the estimates (1.4)

$$|F'(w)| = |f'(z(w))| \frac{1}{|w'(z)|} = \frac{h_1}{\pi} |\operatorname{grad} S|\, |w^2 - 1|^{-1/2} = o\left(\frac{1}{w}\right) = o(1)$$
$$(w \to \infty, \ \operatorname{Im} w \geqslant 0)$$

By the Keldysh-Sedov method we hence obtain the formula /1/

$$F'(w)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\operatorname{Im}(F'(\lambda))}{\lambda-w}\,d\lambda \quad (\operatorname{Im} w>0)$$

Partitioning the interval of integration into the intervals $]-\infty,-1[$, $]-1,1[$ and $]1,\infty[$, using the boundary conditions in (1.8), and integrating in each of the integrals obtained by an appropriate change of variable, we arrive at the identity

$$F'(w)=-\frac{1}{\pi\mu_1}\int_0^{h_1}\frac{\sigma_1(t)}{w+\cos t_1}\,dt-\frac{1}{\pi}\int_0^{\infty}\frac{\varphi(t)}{w-\operatorname{ch} t_1}\,dt \quad (\operatorname{Im} w>0) \tag{2.1}$$

$$t_1=\pi t/h_1$$

after which we find, by integrating with respect to $w$ in the upper half-plane,

$$F(w)=-\frac{1}{\pi\mu_1}\int_0^{h_1}\ln(w+\cos t_1)\,\sigma_1(t)\,dt-\frac{1}{\pi}\int_0^{\infty}\ln(w-\operatorname{ch} t_1)\,\varphi(t)\,dt+ \tag{2.2}$$

$$C\Big(\int_0^{h_1}\sigma_1(t)\,dt+\mu_1\int_0^{\infty}\varphi(t)\,dt\Big) \quad (\operatorname{Im} w>0)$$

where $C$ is an arbitrary constant and any branch regular in the half-plane $\operatorname{Im} w>0$ is taken for the logarithms.

But we find (external normal) from the boundary conditions in (1.7) and the estimates (1.8) taking the estimate (1.5) and the condition of continuity of the stress into account according to the well-known property of harmonic functions

$$\int_0^{h_1}\sigma_1(t)\,dt+\mu_1\int_0^{\infty}\varphi(t)\,dt=\mu_1\int_{\partial\Pi_1^+}\frac{\partial S}{\partial n}\,dl=0$$

Consequently, taking $C=-\ln 2/(\pi\mu_1)$ for convenience and substituting $w=w(z)$ from (2.1) and (2.2), we obtain

$$S(x,y)=\operatorname{Re} F(w(z))=W^{(1)}(x,y)=-\frac{1}{\pi\mu_1}\int_0^{h_1}\ln|2(\cos t_1-\cos z_1)|\,\sigma_1(t)\,dt- \tag{2.3}$$

$$\frac{1}{\pi}\int_0^{\infty}\ln|2(\operatorname{ch} t_1+\cos z_1)|\,\varphi(t)\,dt$$

$$F'(w(z))\,w'(z)=\frac{\partial W^{(1)}}{\partial x}-i\frac{\partial W^{(1)}}{\partial y}=$$

$$\frac{1}{h_1}\sin z_1\Big[-\frac{1}{\mu_1}\int_0^{h_1}\frac{\sigma_1(t)}{\cos t-\cos z_1}\,dt+\int_0^{\infty}\frac{\varphi(t)}{\operatorname{ch} t_1+\cos z_1}\,dt\Big] \tag{2.4}$$

$$t_1=\pi t/h_1,\ z_1=\pi z/h_1$$

Therefore, if a solution $W^{(1)}(x,y)$ exists for problem (1.7) and (1.8), then it is given by (2.3) and its first partial derivatives are determined from (2.4). On the other hand, as is seen from these formulas, the function $W^{(1)}(x,y)$ is obviously harmonic in the half-strip $\Pi_1^+$ and is continuous together with the first partial derivatives up to the boundary $\partial\Pi_1^+$ of this half-strip everywhere with the exception (for the derivatives) of the point $(h_1-0,+0)$. Passage to the limit in (2.4) to the boundary $\partial\Pi_1^+$ of the half-strip $\Pi_1^+$ shows that $W^{(1)}(x,y)$ satisfies the boundary conditions in (1.7). Here

$$\frac{\partial W^{(1)}}{\partial y}(h_1-0,y)=\frac{1}{h_1}\operatorname{sh} y_1\Big(\int_0^{\infty}\frac{\varphi(t)-\varphi(y)}{\operatorname{ch} t_1-\operatorname{ch} y_1}\,dt-\frac{1}{\mu_1}\int_0^{h_1}\frac{\sigma_1(h_1-t)\,dt}{\operatorname{ch} y_1-\cos t_1}\Big)- \tag{2.5}$$

$$\frac{y}{h_1}\varphi(y),\quad y_1=\frac{\pi y}{h_1}$$

The asymptotic formulas

$$h_1 W^{(1)}(x, y) = -\frac{y}{\mu_1}\Big(\int_0^{h_1} \sigma_1(t)\,dt + \mu_1 \int_0^\infty \varphi(t)\,dt\Big) +$$
$$\int_0^\infty t\varphi(t)\,dt - \int_y^\infty (t-y)\,\varphi(t)\,dt + o(1)$$
$$h_1\Big(\frac{\partial W^{(1)}}{\partial x} - i\,\frac{\partial W^{(1)}}{\partial y}\Big) = \frac{1}{i\mu_1}\Big(\int_0^{h_1} \sigma_1(t)\,dt + \mu_1 \int_0^\infty \varphi(t)\,dt\Big) + o(1)$$
$$((x, y) \to \infty,\ (x, y) \in \Pi_1^+)$$

result from these same relations (2.3) and (2.4). Therefore, with the additional conditions

$$\int_0^{h_1} \sigma_1(t)\,dt + \mu_1 \int_0^\infty \varphi(t)\,dt = 0, \quad \int_0^\infty t\varphi(t)\,dt = 0 \tag{2.6}$$

and for absolute convergence of the second integral in (2.6) the solution of the boundary value Problem (1.7) and (1.8) exists, is unique, and is given by (2.3).

*3. Solution of the boundary value Problem (1.9) and (1.10).* Let $S(x, y)$ be some solution of this problem. By virtue of the assumptions made about the continuity of its partial derivatives together with the last of the boundary conditions in (1.9), we will have

$$\frac{\partial S}{\partial x}(x, +0) = 0 \quad (h_1 < x \leqslant H) \tag{3.1}$$

Replacing the condition mentioned in (1.9) by (3.1), we solve the boundary value problem obtained by the Keldysh-Sedov method /1/. As in Sect.2, we introduce the function

$$\begin{gathered} f(z) = S(x, y) + iT(x, y) \in A(\Pi_2^+) \ (z = x + iy) \\ z_2 = \pi(z - h_1)/h_2,\ w(z) = \cos z_2,\ F(w) = f(z(w)) \end{gathered} \tag{3.2}$$

The function $w = w(z)$ is univalent and maps the domain $\Pi_2^+$ of the complex plane $C_z$ conformally into the half-plane $\operatorname{Im} w > 0$ of the complex plane $C_w$; $z = z(w)$ is the appropriate inverse function.

Furthermore, we consider the function

$$g(w) \in \{\sqrt{w^2 - 1}\};\ g(w) \in A\ (w \notin [-1, 1]);\ g(w) \sim w\ (w \to \infty) \tag{3.3}$$

whose existence and uniqueness are obvious /2/. As in Sect.2 we have

$$g(w)\,F'(w) = o(1) \quad (w \to \infty,\ \operatorname{Im} w > 0)$$

from which

$$F'(w) = \frac{1}{\pi g(w)} \int_{-\infty+i0}^{\infty+i0} \frac{\operatorname{Im}[F'(\lambda)\,g(\lambda)]}{\lambda - w}\,d\lambda \quad (\operatorname{Im} w > 0) \tag{3.4}$$

Calculating the value of $g(\lambda)$ in the intervals $-\infty < \lambda < -1$ and $1 < \lambda < \infty$, the value of $g(\lambda + i0)$ in the interval $-1 < \lambda < 1$, using the first two boundary conditions in (1.9) and the relationship (3.1) and making the change in the variable of integration $\lambda = -\operatorname{ch}(\pi t/h_2)$, we obtain

$$F'(w) = -\frac{\omega}{\pi g(w)} \int_0^\infty \frac{\varphi(t)\operatorname{sh} t_2}{w + \operatorname{ch} t_2}\,dt \quad (\operatorname{Im} w > 0), \quad t_2 = \frac{\pi t}{h_2} \tag{3.5}$$

Integrating in the upper half-plane between $\infty$ and $w$, we obtain

$$F(w) = \frac{\omega}{\pi} \int_0^\infty \ln\Big(\frac{w + g(w) + \exp t_2}{w + g(w) + \exp(-t_2)}\Big)\varphi(t)\,dt \quad (\operatorname{Im} w > 0) \tag{3.6}$$

Here the regular branch of the integrand in the half-plane $\operatorname{Im} w > 0$ is fixed by the condition of vanishing at infinity.

It is seen that the selection of the regular branch $g(w)$ of the analytic function $\{\sqrt{w^2-1}\}$ according to the method mentioned in (3.3) will be

$$g(w(z)) = g(-\cos z_2) = i \sin z_2 \quad (z \in \Pi_2^+)$$

According to (3.2), (3.6), and (3.5)

$$S(x, y) = \operatorname{Re} F(w(z)) = W^{(2)}(x, y) = \frac{\omega}{\pi} \int_0^\infty \ln \left| \frac{\exp t_2 - \exp(-iz_2)}{\exp(-t_2) - \exp(-iz_2)} \right| \varphi(t)\, dt \tag{3.7}$$

$$F'(w(z))\, w'(z) = \frac{\partial W^{(2)}}{\partial x} - i \frac{\partial W^{(2)}}{\partial y} = \frac{i\omega}{h_2} \int_0^\infty \frac{\varphi(t) \operatorname{sh} t_2}{\operatorname{ch} t_2 - \cos z_2}\, dt \tag{3.8}$$

Thus, if a solution $W^{(2)}(x, y)$ exists for the boundary value Problem (1.9) and (1.10), then it is unique, is given by (3.7), and its first partial derivatives are determined from the equalities (3.8). On the other hand, this function is obviously harmonic in $\Pi_2^+$ and satisfies the conditions of continuity mentioned in Sect.1. Satisfaction of the first two boundary conditions in (1.9) is confirmed by passing to the limit to $\partial\Pi_2^+$ in identity (3.8). Here $(y > 0)$

$$\frac{\partial W^{(2)}}{\partial y}(h_1 + 0, y) = -\frac{\omega}{h_2} \int_0^\infty \frac{\varphi(t) \operatorname{sh} t_2 - \varphi(y) \operatorname{sh} y_2}{\operatorname{ch} t_2 - \operatorname{ch} y_2}\, dt + \frac{\omega}{h_2} y \varphi(y), \quad y_2 = \frac{\pi y}{h_2} \tag{3.9}$$

Using (1.5), (3.7) and (3.8), it is easy to see that the estimate (1.10) also holds. Finally, setting $z_2 = \pi(x - h_1)/h_2$ in (3.7), we find

$$W^{(2)}(x, 0) = \frac{\omega}{h_2} \int_0^\infty t \varphi(t)\, dt \quad (h_1 \leqslant x \leqslant H)$$

Therefore, under the additional condition that the second of the equalities (2.6) holds, the solution of the boundary value Problem (1.9) and (1.10) exists, is unique, and is given by (3.7), and its first partial derivatives are determined from the equalities (3.8). We hence also obtain from (1.3) an expression of the objective function $\sigma(x)$ in terms of the auxiliary unknown functions $\varphi(y)$

$$\sigma(x) = -\frac{\mu_1}{h_2} \int_0^\infty \frac{\varphi(t) \operatorname{sh} t_2}{\operatorname{ch} t_2 - \cos x_2}\, dt \quad (h_1 < x < H), \quad x_2 = \pi(x - h_1)/h_2 \tag{3.10}$$

***4. The integral equation for the auxiliary unknown function. The characteristic index. The asymptotic form of the objective function. Concluding remarks.*** Differentiating the identity (1.1) and using (2.5) and (3.9), we obtain the following integral equation for the function $\varphi(y)$

$$\begin{gathered} \frac{1}{h_1} \operatorname{sh} y_1 \int_0^\infty \frac{\varphi(t) - \varphi(y)}{\operatorname{ch} t_1 - \operatorname{ch} y_1}\, dt + \frac{\omega}{h_2} \int_0^\infty \frac{\varphi(t) \operatorname{sh} t_2 - \varphi(y) \operatorname{sh} y_2}{\operatorname{ch} t_2 - \operatorname{ch} y_2}\, dt - \\ \left(\frac{1}{h_1} + \frac{\omega}{h_2}\right) y \varphi(y) = \frac{1}{\mu_1 h_1} \operatorname{sh} y_1 \int_0^{h_1} \frac{\sigma_1(h_1 - t)}{\operatorname{ch} y_1 - \cos t_1}\, dt \quad (0 < y < \infty) \end{gathered} \tag{4.1}$$

under the additional constraints (2.6). The mathematical meaning of these constraints is noted above for each of the auxiliary boundary value problems. The mechanical meaning is naturally formulated in terms of the whole bielectric strip $\Pi = \Pi_1 \cup \Pi_2$. From (2.6) and (3.10) we have

$$\int_0^{h_1} (-\sigma_1(x))\, dx + \int_{h_1}^H \sigma(x)\, dx = 0$$

In combination with the boundary conditions in (1.7) and (1.9), that do not contain $\varphi(y)$, and the conditions at infinity (1.8) and (1.10), this yields

$$\int_{\partial\Pi^+} \frac{\partial W}{\partial n}\, dl = 0, \quad W = \begin{cases} W^{(1)}(x, y), & (x, y) \in \Pi_1^+ \\ W^{(2)}(x, y), & (x, y) \in \Pi_2^+ \end{cases}$$

In other words the bielastic half-strip $\Pi^+ = \Pi_1^+ \cup \Pi_2^+$ and also the half-strip $\Pi^-$ by symmetry, are self-equilibrated systems.

Analogously, we deduce from (2.6) and (3.7)

$$\int_0^{h_1} W^{(1)}(x, \pm 0)\, dx = 0$$

i.e., the crack $\gamma$ in the bielastic strip $\Pi$ does not experience displacement as a whole. Taking account of the last boundary condition in (1.9), we can assert the very same about the plane of symmetry of the whole system $\Pi$.

Now, from the prescribed general form of the asymptotic form (1.5) of the function $\varphi(y)$ as $y \to +0$, we find the characteristic index $\alpha$ and the conditions on the coefficients $a_0, b_0, a_1$ by using relations (4.1) and (2.6). Let us use the notation $\Phi(y), \Psi(y), \Pi(y)$ for the components with integrals in (4.1) in order of their succession. Applying the elementary method of asymptotic estimates of the definite integrals /3, 4/, we obtain

$$\Phi(y) = -a_0\left(\operatorname{ctg}\frac{\pi\alpha}{2}\right) y^{\alpha-1} + \left(\frac{a_0}{h_1} + a_1 \operatorname{tg}\frac{\pi\alpha}{2}\right) y^\alpha + O(y^\gamma)$$

$$\frac{1}{\omega}\Psi(y) = a_0\left(\operatorname{tg}\frac{\pi\alpha}{2}\right) y^{\alpha-1} + \frac{2}{\pi} b_0 \ln\frac{1}{y} + C(h_2, \varphi) +$$

$$\left(\frac{a_0}{h_2} - a_1 \operatorname{ctg}\frac{\pi\alpha}{2}\right) y^\alpha + O(y^\gamma) \quad (y \to +0), \quad \gamma = \frac{\alpha+1}{2}$$

with the constant $C(h_2, \varphi)$, dependent on $h_1$ in terms of the function $\varphi(y)$. Imposing the additional condition of differentiability at the point $x = h_1 - 0$ on the function $\sigma_1(x)$, we will have

$$\Pi(y) = \sigma_1(h_1) + O(y \ln y) = \sigma_1(h_1) + O(y^\gamma) \quad (y \to +0)$$

Finally, we find directly from expansion (1.5)

$$\left(\frac{1}{h_1} + \frac{\omega}{h_2}\right) y\varphi(y) = a_0\left(\frac{1}{h_1} + \frac{\omega}{h_2}\right) y^\alpha + O(y^\gamma) \quad (y \to +0)$$

Substituting these expressions into the identity (4.1), we arrive at the relationships

$$a_0\left(-\operatorname{ctg}\frac{\pi\alpha}{2} + \omega \operatorname{tg}\frac{\pi\alpha}{2}\right) = 0, \quad b_0 = 0$$
$$C(h_2, \varphi) = \frac{\mu_2}{\mu_1^2}\sigma_1(h_1), \quad a_1\left(\operatorname{tg}\frac{\pi\alpha}{2} - \omega \operatorname{ctg}\frac{\pi\alpha}{2}\right) = 0 \tag{4.2}$$

On the other hand, we obtain from (3.10) by the same procedures of elementary asymptotic estimates

$$\sigma(x) = K_0 (x - h_1)^{\alpha-1} - K_1 + K_2 (x - h_1)^\alpha + O((x - h_1)^\gamma) \tag{4.3}$$
$$(x \to h_1 + 0)$$
$$K_0 = -\mu_1 a_0/\cos {}^1/_2\, \pi\alpha, \quad K_1 = \mu_1 C(h_2, \varphi), \quad K_2 = -\mu_1 a_1/\sin {}^1/_2 \pi\alpha$$

It is well-known that for $\mu_1 = \mu_2$, i.e., for $\alpha = {}^1/_2$, the stress intensity factor $k_{\mathrm{III}} = \sqrt{2\pi} K_0$ is different from zero in the case of a uniformly distributed load $\sigma_1(x) = \tau =$ const. The integral Eq.(4.1) depends continuously on $\sigma_1(x)$ and $\omega = \mu_1/\mu_2$. Consequently, it can be assumed that $a_0 = 0$ generally. In this case and under the condition $\alpha > 0$ we will have from (4.2) and (4.3) $(\mu_1 \neq \mu_2)$

$$\alpha = 2\pi^{-1} \operatorname{arctg} \sqrt{1/\omega}\ (\omega = \mu_1/\mu_2), \quad \varphi(y) = a_0 y^{\alpha-1} + O(y^{(\alpha+1)/2}) \quad (y \to +0)$$
$$\sigma(x) = -(\mu_1 a_0/\cos {}^1/_2 \pi\alpha)\ (x - h_1)^{\alpha-1} - \sigma_1(h_1)/\omega + O((x - h_1)^{(\alpha+1)/2})$$
$$(x \to h_1 - 0)$$

For $\mu_1 = \mu_2$ the estimates of the residual terms are worsened: the appropriate index will be $^1/_2$.

Thus, the stress intensity factor $k_{III} = -\mu_1\sqrt{2\pi}a_0/\cos{}^1/_2\pi\alpha$ is expressed explicitly in terms of the coefficient $a_0$ of the expansion (1.5), because the characteristic index $\alpha$ is now known. In its turn the quantity $a_0$ can be found for a specific set of parameters by using a numerical solution of (4.1) under the Condition (2.6). Specific difficulties occur only for $h_2 \gg h_1$ and especially for $h_2 \ll h_1$. In these situations approximate formulas for $k_{III}$ would be useful.

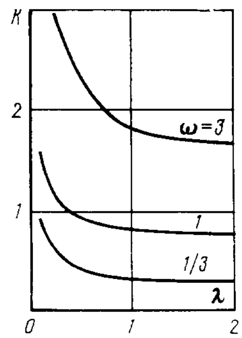


Fig.2

Some results of a numerical computation are shown in Fig.2 where $\sigma_1(x) = \tau = \text{const}$, $K = k_{III}/\sqrt{2\pi}h_1^{1-\alpha}\tau$, $\lambda = h_2/h_1$. The mean of these curves agrees with less than 1% error with the graph of the dependence for a homogeneous medium /6/.

Note that the characteristic index $\alpha$ turns out to be independent of either the monolayer thicknesses $h_1, h_2$ or of the specific form of the continuous load $\sigma_1(x)$. It is identical here with the appropriate quantity obtained earlier /7/ in the homogeneous problem with $h_1 = h_2 = \infty$. It should just be emphasized that this agreement is essentially related to the assumption of the boundedness of the stress on the crack near its tip.

The problem of an edge crack in a bielastic strip (or equivalently of a central crack in a symmetric trielastic strip) was examined in /8-11/. Plane strain was studied in /9/. It was assumed in /8, 10/ that the tips of the central longitudinal shear crack are located strictly within a homogeneous medium so that the characteristic index $\alpha = {}^1/_2$ (here $h_2 = \infty$). The results obtained in this paper were partially presented in the note /11/.

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